Linear phase Filtering

$$e^{j\omega n} \longrightarrow \boxed{\begin{matrix}LTI\\ h(n)\end{matrix}} \longrightarrow e^{j\omega n} H(\omega)$$

$H(\omega)$: DTFT of $h(n)$, Freq. Response filter at ω

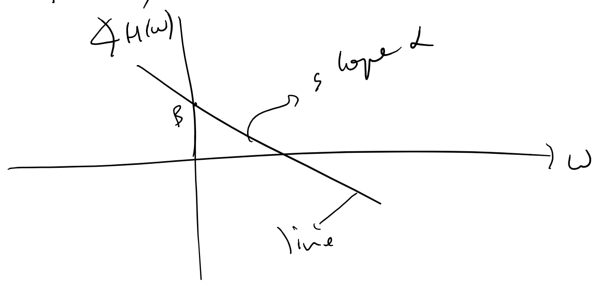
$$x(n) \longrightarrow \boxed{\begin{matrix}LTI\\ h(n)\end{matrix}} \longrightarrow y(n)$$

$$Y(\omega) = X(\omega)\, H(\omega) \qquad H(\omega) = |H(\omega)|\, e^{j\measuredangle H(\omega)}$$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$

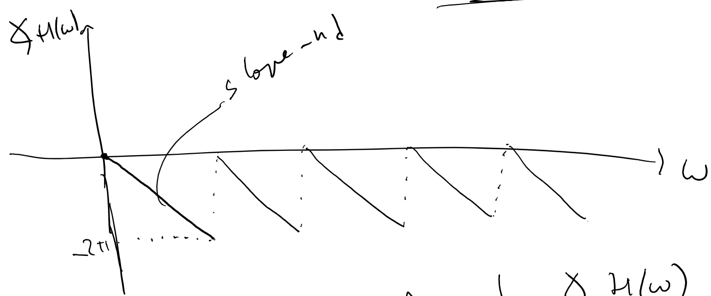
Def Linear phase LTI system: if $\measuredangle H(\omega)$ is linear in ω



Consider a pure delay LTI system

$$h(n) = \delta(n - n_d)$$

$$|H(\omega)| = 1 \qquad \measuredangle H(\omega) = -\omega n_d$$



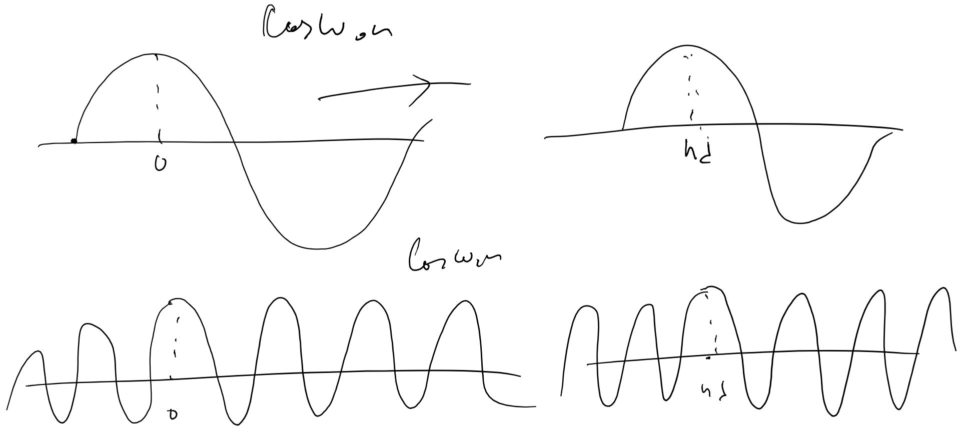
Def: Group delay $\triangleq -\dfrac{d}{d\omega} \measuredangle H(\omega)$

$$\measuredangle H(\omega) = -\omega n_d \Rightarrow G.D = -\frac{d}{d\omega}(-\omega n_d) = n_d$$

intuitively n_d is the amount of delay for ALL frequencies; is independent of ω.

⟹ All frequencies are delayed by the same amount as they pass through the system.

$\cos \omega_0 n + \cos \omega_1 n \rightarrow \boxed{h(n) = \delta(n-n_d) \quad \text{G.D. } n_d}$

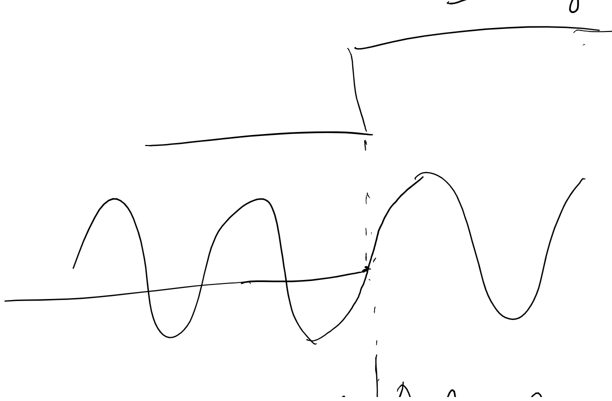


Observation: If Linear phase ⟹ group delay is constant.

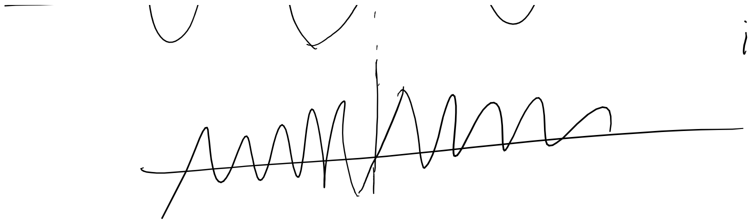
⟹ relative phase of sinusoidal components of a signal remain intact as they go through the system.

→ no distortion added.

Image Processing



To preserve edges in image processing it is good to have



it is good to have linear phase

Consider a narrow band signal.

$\omega_0 = 900\,\text{MHz}$

$\Delta\omega = 5\,\text{kHz}$

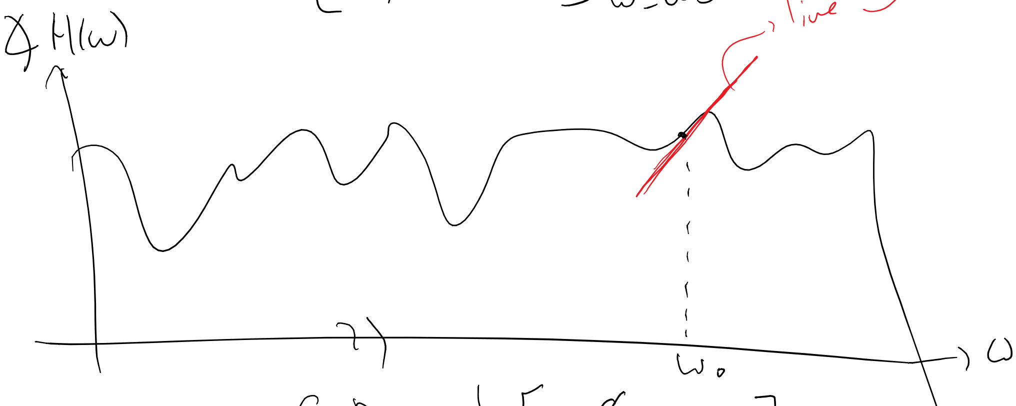
$\Delta\omega \ll \omega_0$

$$\boxed{x(n) = s(n)\cos\omega_0 n}$$

Pass $x(n)$ Through LTI filter $H(\omega)$ s.th. $|H(\omega)| = 1$. Can approximate $\angle H(\omega)$ at $\omega = \omega_0$ with a linear Term

$$\left[\angle H(\omega)\right]_{\omega=\omega_0} = -\phi - \omega n_d$$

line



$$G.D = -\frac{d}{d\omega}\left[-\phi - \omega n_d\right] = n_d$$

Then: $x(n) \rightarrow \boxed{H(\omega)} \rightarrow y(n)$

Can show $y(n) = s(n - n_d)\cos(\omega_0 n - \omega_0 n_d - \phi)$

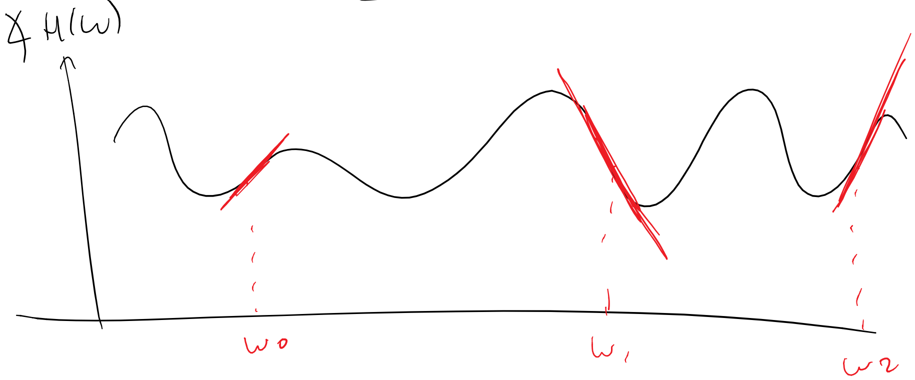
Conclusion: For a narrowband signal content around ω_0 delay is proportional

To $\quad GD = \left[-\frac{d}{d\omega} \measuredangle H(\omega) \right]_{\omega=\omega_0}$

ω_0 = center frequency of narrowband signal

$x(n) = a \cos \omega_0 n + b \cos \omega_1 n + c \cos \omega_2 n$

$$x(n) \longrightarrow \boxed{\begin{matrix} h(n) \\ H(\omega) \end{matrix}} \longrightarrow y(n)$$



$a \cos \omega_0 n \longrightarrow a \cos(\omega_0 n - \omega_0 n_{d_0} - \phi_0)$

$b \cos \omega_1 n \longrightarrow b \cos(\omega_1 n - \omega_1 n_{d_1} - \phi_1)$

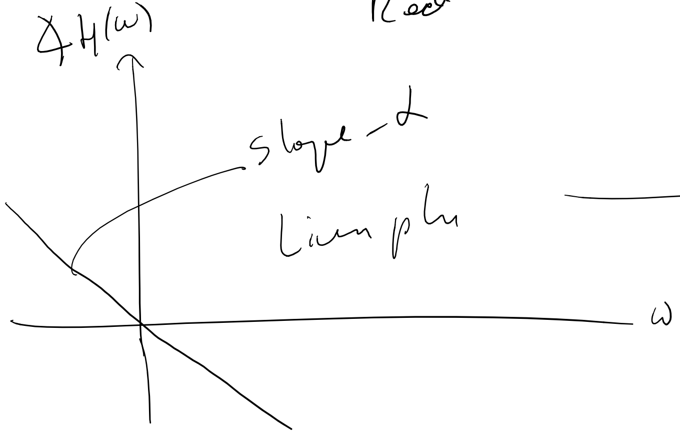
$c \cos \omega_2 n \longrightarrow c \cos(\omega_2 n - \omega_2 n_{d_2} - \phi_2)$

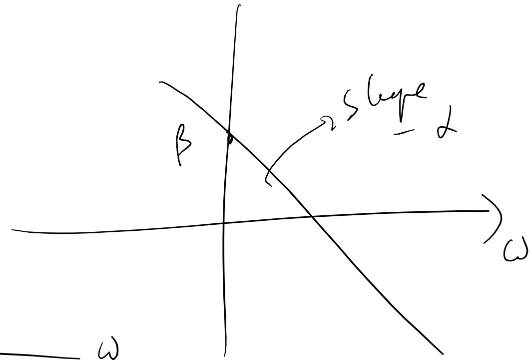
Def $\quad H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} e^{-j\alpha\omega}$ $\quad$ linear phase

Generalized Linear Phase:

$H(\omega) = H_m(\omega) \, e^{j(\beta - \alpha\omega)}$ $\quad$ generalized linear phase

Real





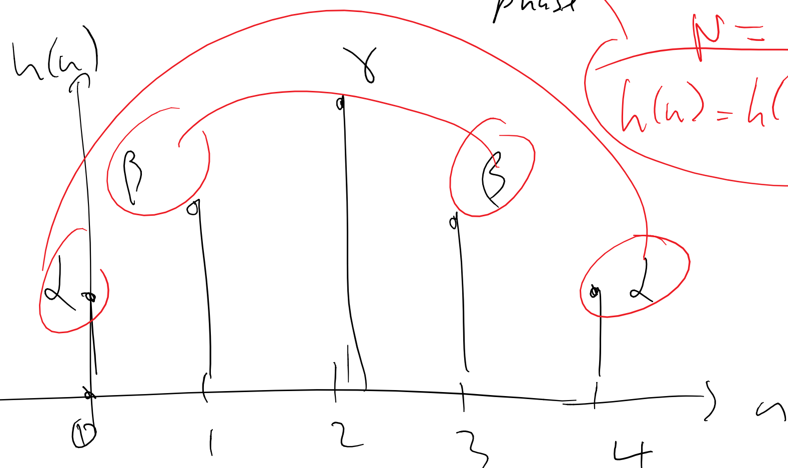
If imposes symmetry in FIR filter
Then → linear phase.

Show: If $h(n) = h(N-1-n)$ for FIR filter with real coeff ⟹ linear phase

N Tap filter

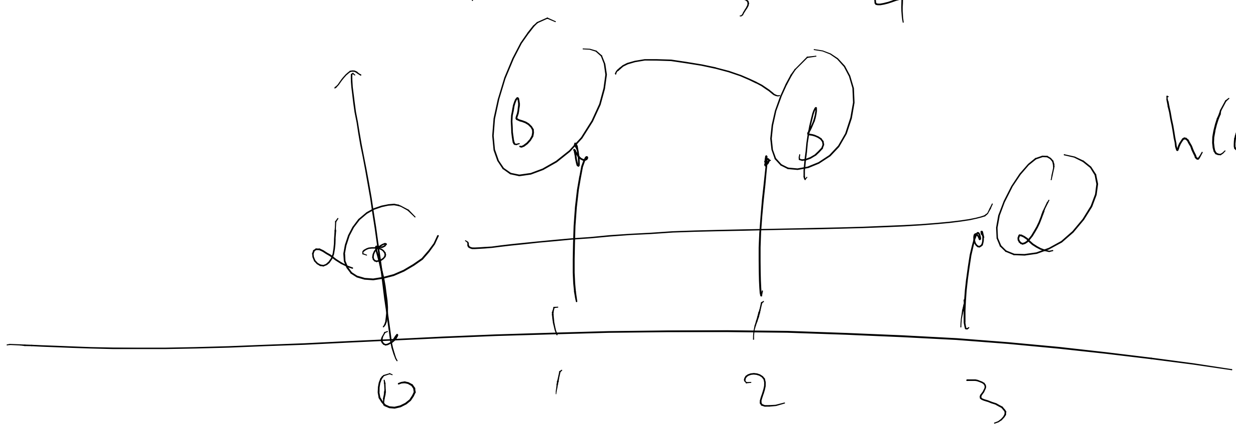
$N = 5$

$h(n) = h(4-n)$



$N = 4$

$h(n) = h(3-n)$



Recall $H(\omega) = \sum^{N-1} h(n)\, e^{-j\omega n}$

Proof: $H(\omega) = \sum_{n=0} h(n) e^{-j}$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n) e^{-j\omega n} + \underbrace{\sum_{n=N/2}^{N-1} h(n) e^{-j\omega n}}_{\text{change of variable}}$$

$m = N-1-n \leftarrow$

$$= \sum_{n=0}^{N/2-1} h(n) e^{-j\omega n} + \sum_{m=0}^{N/2-1} h(N-1-m) e^{-j\omega(N-1-m)}$$

Impose Assumption $h(n) = h(N-1-n)$

$$= \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n} + e^{-j\omega(N-1-n)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n} e^{j\omega\frac{(N-1)}{2}} + e^{j\omega n} e^{-j\omega\frac{(N-1)}{2}} \right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \underbrace{\sum_{n=0}^{N/2-1} h(n) \left[2\cos\left(\omega n - \frac{\omega}{2}(N-1)\right) \right]}_{\text{Real } H_m(\omega)}$$

$\alpha = \frac{N-1}{2}$

$$H(\omega) = H_m(\omega) e^{-j\alpha\omega}$$

$H(\omega) = H_m(\omega) e^{\ldots}$

$\Longrightarrow$ linear phase.

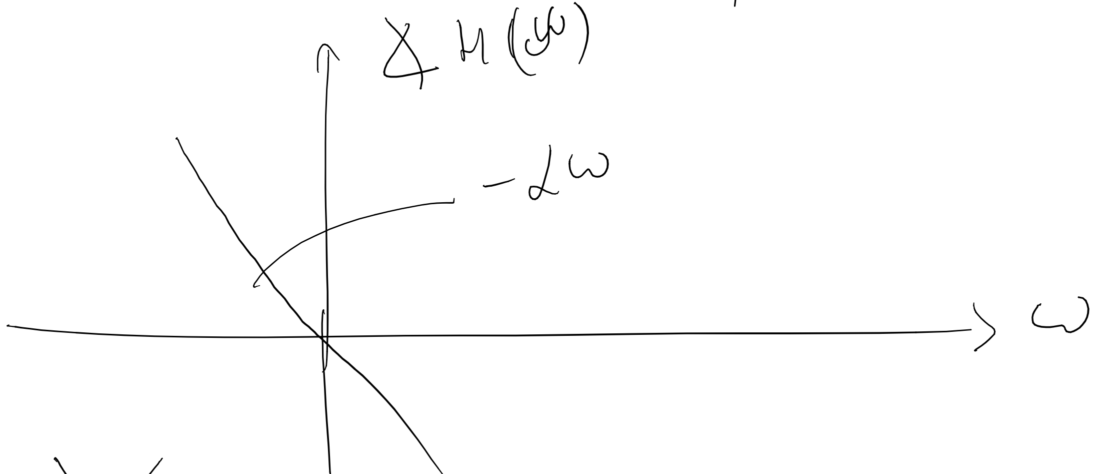
Q: what is $\measuredangle H(\omega)$

$$H(\omega) = |H(\omega)| e^{j \measuredangle H(\omega)} = H_m(\omega) e^{-j\alpha\omega}$$

Consider 2 Cases:

① $H_m(\omega) > 0$ positive $\Longrightarrow \measuredangle H(\omega) = e^{-j\alpha\omega}$

$\Longrightarrow H_m(\omega) = |H(\omega)|$



② $H_m(\omega) < 0 \longrightarrow$ negative

$$H(\omega) = |H_m(\omega)| (-1) e^{-j\alpha\omega}$$

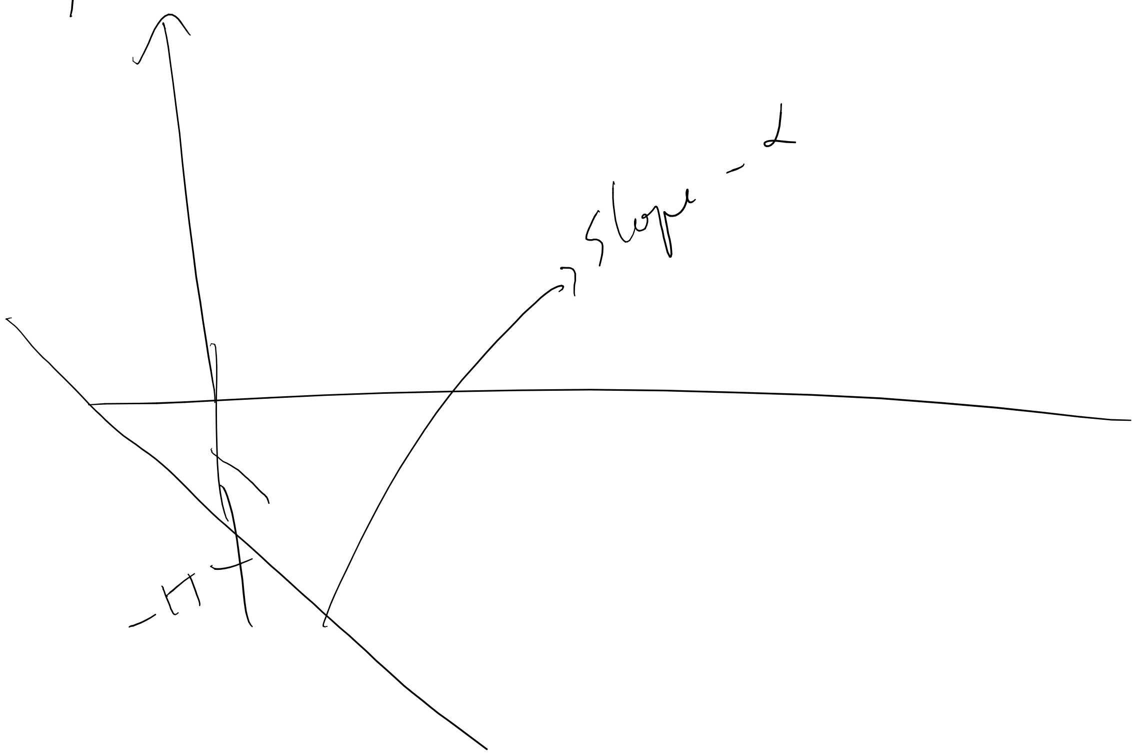
$$= |H_m(\omega)| e^{-j\pi} e^{-j\alpha\omega}$$

$$= |H_m(\omega)| e^{-j(\alpha\omega + \pi)} \Bigg\} \Longrightarrow \measuredangle H(\omega)$$

$\measuredangle H(\omega)$

$$= |H_m(\omega)|\, e$$

$$\measuredangle H(\omega) \quad = |H(\omega)|\, e^{j\measuredangle H(\omega)} \quad \Bigg\} \Longrightarrow \measuredangle H(\omega) = -\alpha\omega - \pi$$



Symmetry in FIR filter can also decrease Multipl. count.

$$x(n) \longrightarrow \boxed{\begin{array}{c} h(n) \\ h(n)=h(N-1-n) \end{array}} \longrightarrow y(n)$$

$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

$$= \sum^{N/2-1} h(k)\, x(n-k) + \sum_{k=N/2}^{N-1} h(k)\, x(n-k)$$

$$= \sum_{k=0} h(k)\, x(n-k) + \sum_{k=N/2}$$

Change variable

$k = N-1-m$

$$y(n) = \sum_{k=0}^{N/2-1} h(k)\, x(n-k) + \sum_{m=0}^{N/2-1} h(N-1-m)\, x(n-N+1+m)$$

$$y(n) = \sum_{k=0}^{N/2-1} h(k) \underbrace{\left[x(n-k) + x(n-N+1+k) \right]}_{w(k)}$$

For each $y(n)$: $N/2$ mults

$N/2$ adds $+$ $N/2$ adds to form $w(k)$

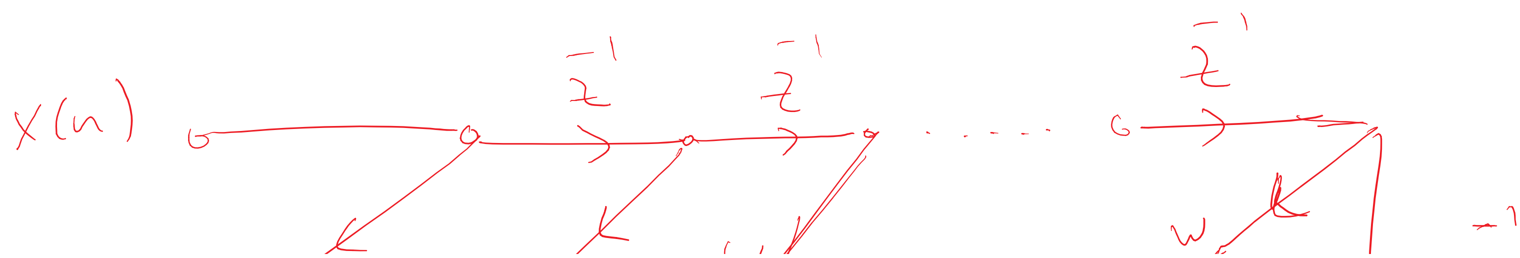
Regular:

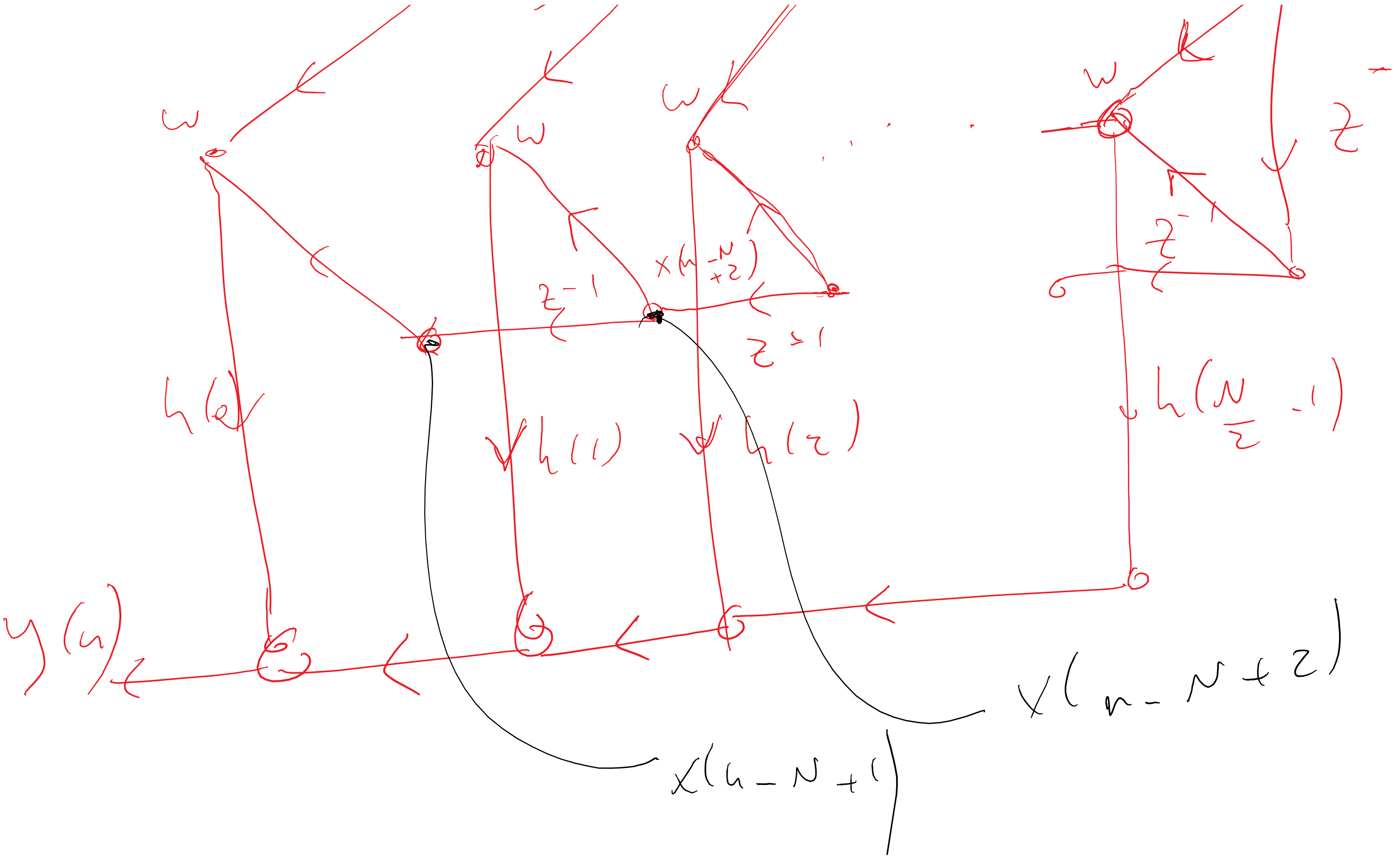
$$\sum h(k)\, x(n-k)$$

N mults

N adds

$1/2$ as many mults





April 11th

Condition for Achieving Linear phase

$$H(\omega) = \underbrace{H_m(\omega)}_{\substack{\text{Real} \\ + \\ \text{or} \\ -}} \, e^{j(\beta - \alpha\omega)}$$

α = group delay

$\measuredangle H(\omega) = \beta - \alpha\omega$

$-\frac{d}{d\omega}(\measuredangle H(\omega)) = \alpha$

$$H(\omega) = H_m(\omega)\cos(\beta - \alpha\omega) + j\,H_m(\omega)\sin(\beta - \alpha\omega)$$

$T \quad \times \quad 11/\ldots$

$\dfrac{\sin(\beta - \alpha\omega)}{} = \tan(\beta - \alpha\omega)$

$$\tan \measuredangle H(\omega) = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \tan(\beta - \alpha\omega)$$ equ 1

Derive $\measuredangle H(\omega)$ in Terms of $h(n)$.

$$H(\omega) = \sum h(n) e^{-j\omega n}$$

$$H(\omega) = \sum_n h(n) \cos \omega n - j \sum_n h(n) \sin(\omega n)$$

$$\tan \measuredangle H(\omega) = \frac{-\sum_n h(n) \sin(\omega n)}{\sum_n h(n) \cos(\omega n)}$$ Eqn 2

Combine 1 & 2 $\Rightarrow$

$$\boxed{\frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \frac{-\sum_n h(n) \sin \omega n}{\sum_n h(n) \cos \omega n}}$$

Necessary Condition for $h(n)$ to be GLP.

$$\Rightarrow \sin(\beta - \alpha\omega) \sum_n h(n) \cos \omega n + \cos(\beta - \alpha\omega) \sum_n h(n) \sin \omega n = 0$$

$$\boxed{\sum_{n=0}^{N-1} h(n) \sin[\omega(n - \alpha) + \beta] = 0}$$

$$\sum_{n=0} h(n) \sin\left[\omega(n-\alpha)+\beta\right] = 0$$

→ Necessary Condition for $h(n)$ To be G.L.P.

Case ①: $\beta = 0$ or π

$$\sum_{n=0}^{N-1} h(n) \sin\left(\omega(n-\alpha)\right) = 0$$

Can show: If $N = 2\alpha + 1$

$h(n) = h(N-1-n)$

Then we satisfy the condition.

Case ②: $\beta = \pi/2$ or $3\pi/2$

$$\sum_{n=0}^{N-1} h(n) \cos\left[(n-\alpha)\omega\right] = 0$$

Can show if $N = 2\alpha + 1$

$h(N-1-n) = -h(n)$

This is satisfied

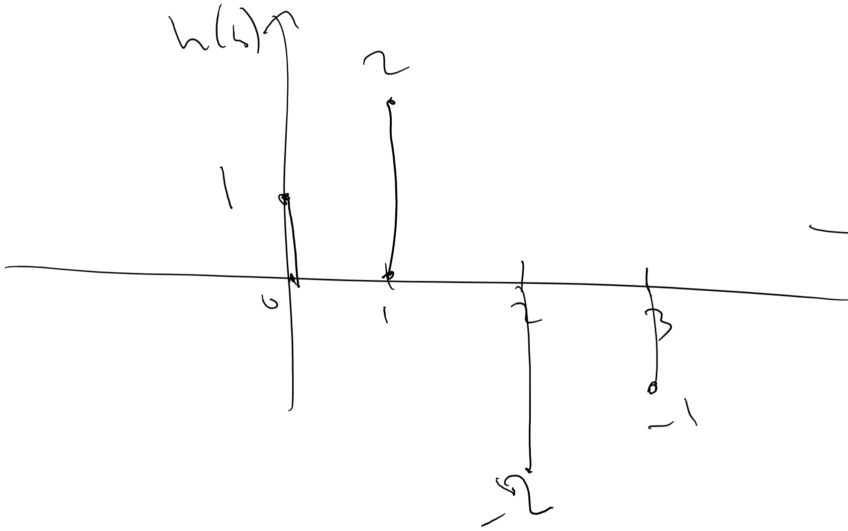
Ex $N = 4$ # of Taps: $h(3-n) = -h(n)$

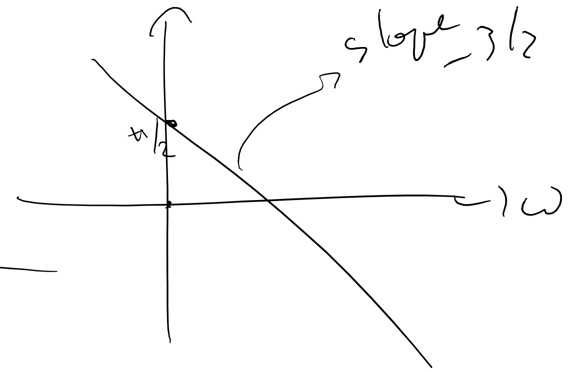
even $\beta = \pi/2$

$$H(\omega) = H_m(\omega)\, e^{j\left(\frac{\pi}{2} - \frac{3}{2}\omega\right)}$$

$N = 2\alpha + 1 \Rightarrow \alpha = 3/2$

$\measuredangle H(\omega)$



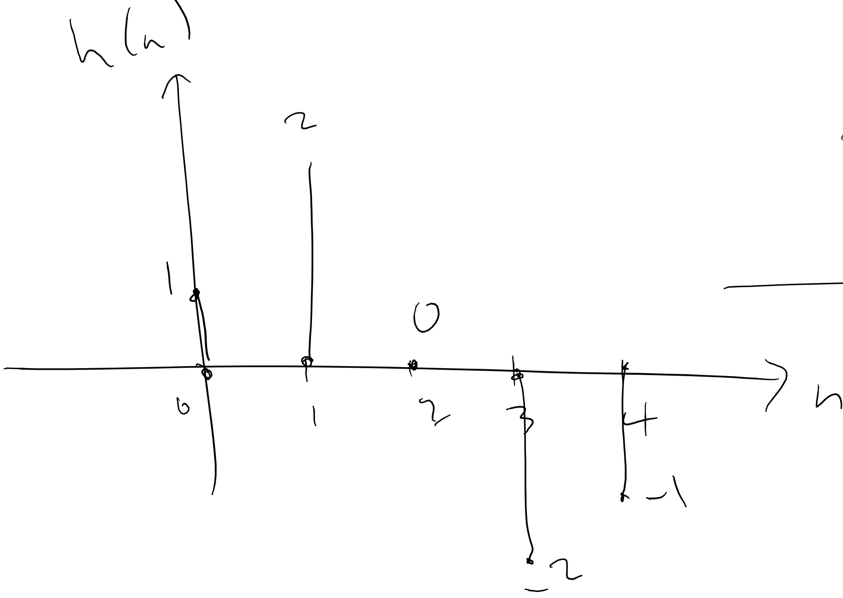


<u>Ex</u> $\quad N = 5$

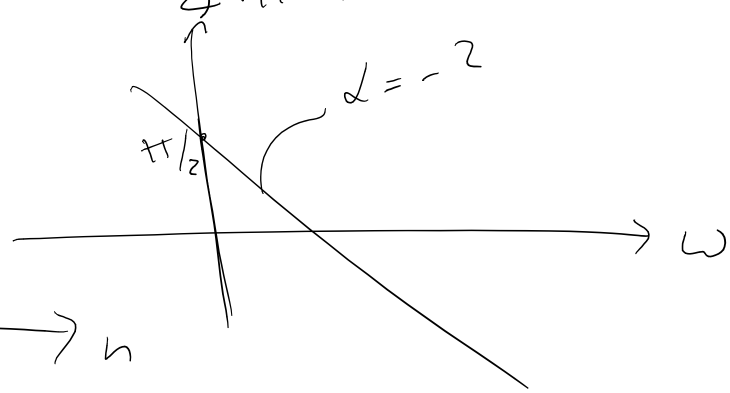
$N = 2\alpha + 1 = 5 \Rightarrow \alpha = 2$

$\beta = \pi/2$

$$h(N-1-n) = -h(n) \Rightarrow h(4-n) = -h(n)$$



$\measuredangle H(\omega)$



Case (1) $\quad \beta = 0 \qquad h(N-1-n) \qquad 0 \le n < N$

$h(n) = \Big\{ \quad 0 \quad$ otherwise

$$h(n) = \begin{cases} 0 & \text{otherwise} \end{cases}$$

$$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} \; e^{j\left(\frac{N-1}{2}\right)\omega}$$

$\beta = 0$

$\alpha = \frac{N-1}{2}$

Case (2) $\beta = \pi/2$

$$h(n) = \begin{cases} -h(N-1-n) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$H(\omega) = H_m(\omega)\, e^{j(\beta - \alpha\omega)}$$

$\beta = \pi/2$

$\alpha = \frac{N-1}{2}$

$$H(\omega) = H_m(\omega)\, e^{j\pi/2}\, e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$H(\omega) = H_m(\omega)\; \underline{e} \quad \underline{e}$$

j

β

Case (1) $\beta = 0$ Symmetry

Case 2 $\beta = \pi/2$ anti-symmetry

odd # of taps — Type I

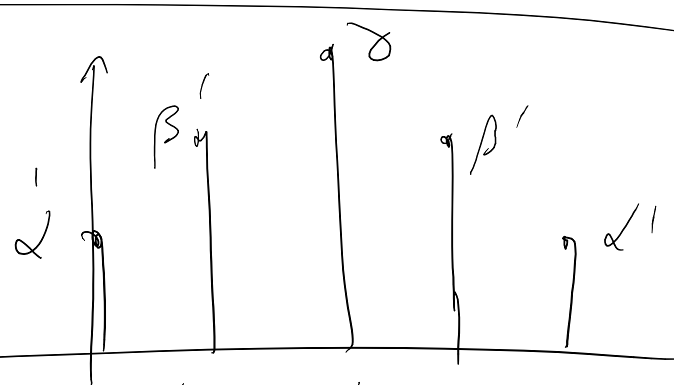
even # of taps — Type II

odd # of taps — Type III

even # taps — Type IV

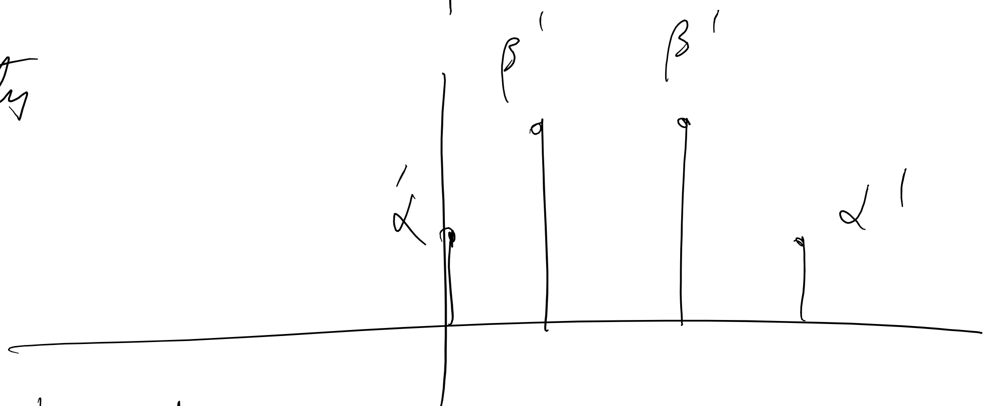
Type I Symmetry

$N = 5$



Type II Symmetry

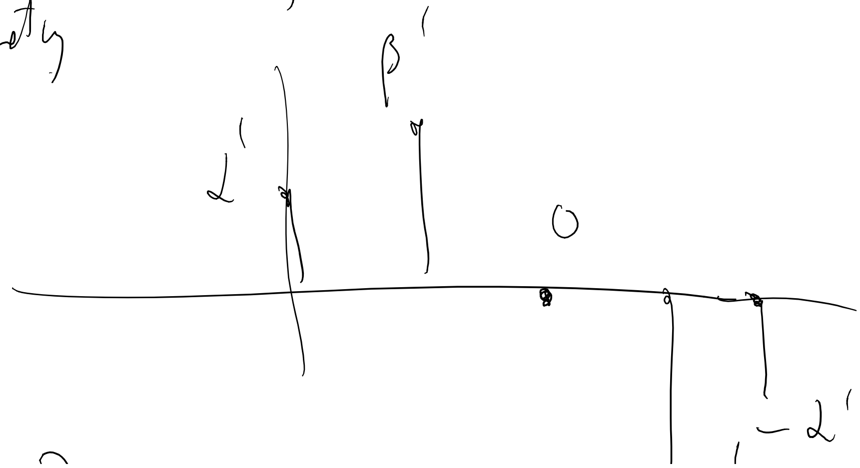
$N = 4$



Type III: Antisymmetry

N odd

$N = 5$

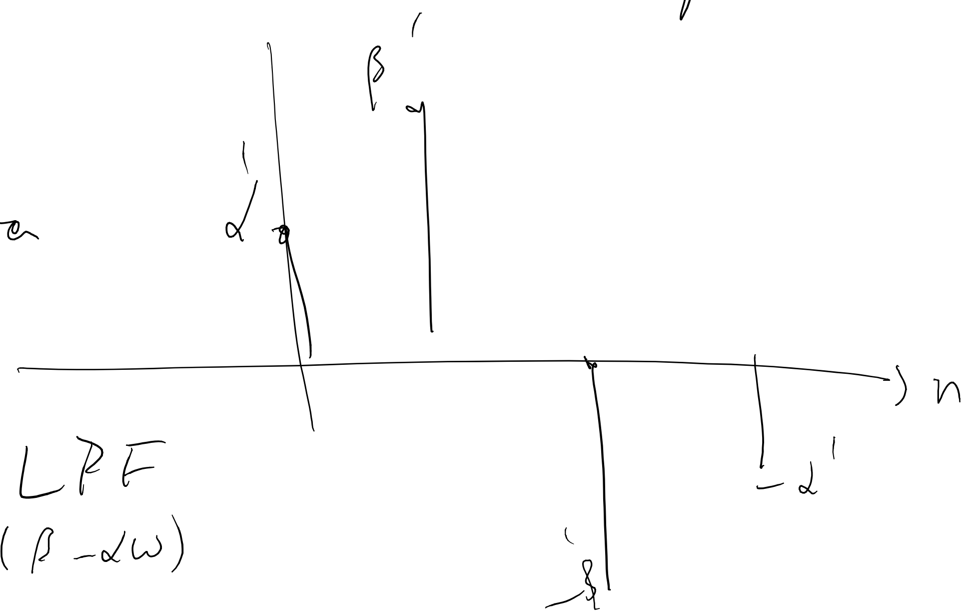


$$\left.\begin{array}{l} H(0) = 0 \\ H(\pi) = 0 \end{array}\right\} \longrightarrow \text{Cannot be LPF, HPF}$$

Can be Band pass.

Type IV
antisymmetry
and N even

$\beta = \pi/2$

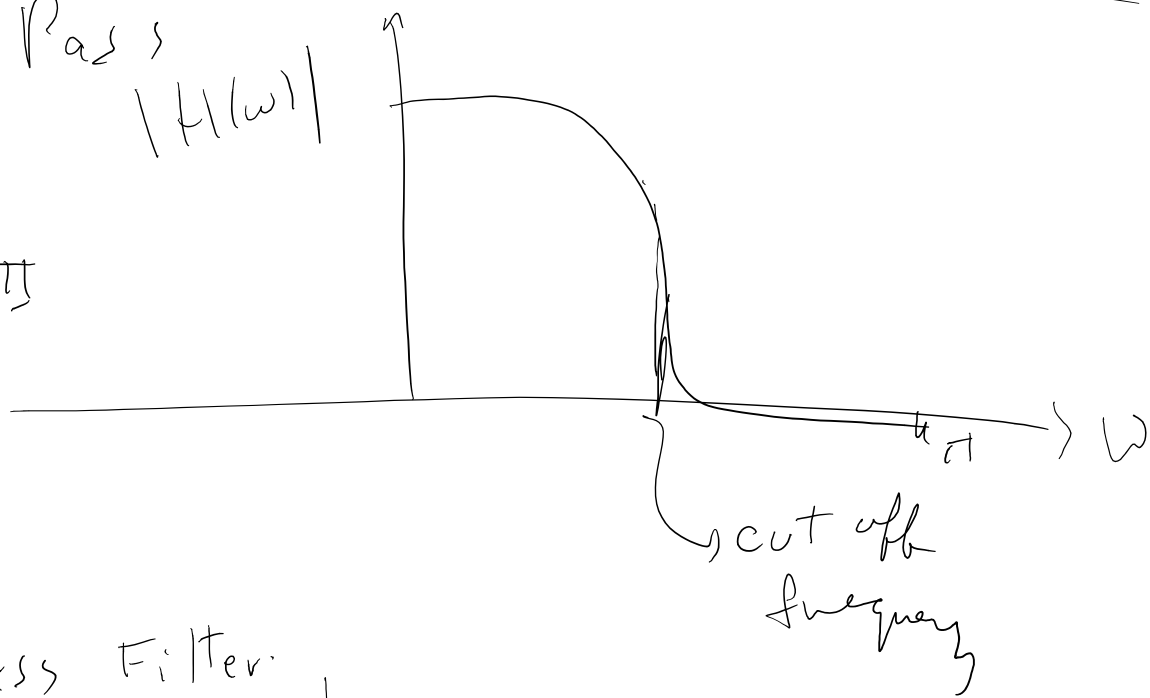


Cannot be LPF

$H(\omega) = H_m(\omega)\, e^{j(\beta - \alpha\omega)}$

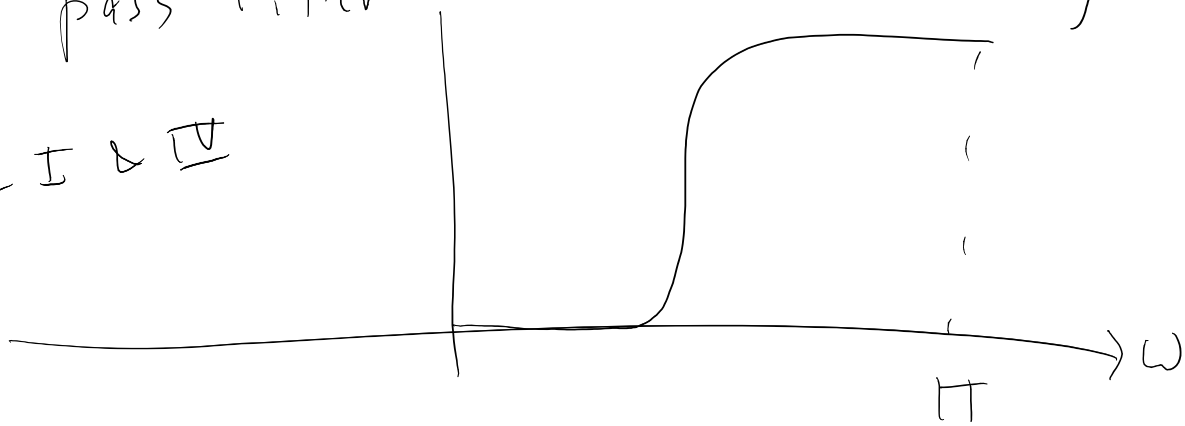
	Symmetry or antisymmetry	N even or odd	α	β	$H_m(\omega)$	Constraints
Type 1	$h(n) = h(N-1-n)$	odd	$\frac{N-1}{2}$	0	$\sum_{0}^{\frac{N-1}{2}} a(n) \cos \omega n$	Real.
II	$h(n) = h(N-1-n)$	even	$\frac{N-1}{2}$	0	$\sum_{n=1}^{N/2} b(n) \cos \omega(n-\frac{1}{2})$	Real. $H(\pi) = 0$
III	$h(n) = -h(N-1-n)$	odd	$\frac{N-1}{2}$	$\pi/2$	$\sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n$	purely imaginary $H(0) = 0$ $H(\pi) = 0$
IV	$h(n) = -h(N-1-n)$	even	$\frac{N-1}{2}$	$\frac{\pi}{2}$	$\sum_{n=1}^{N/2} d(n) \sin \omega(n-\frac{1}{2})$	purely imaginary $H(0) = 0$

Low Pass



Type I & II

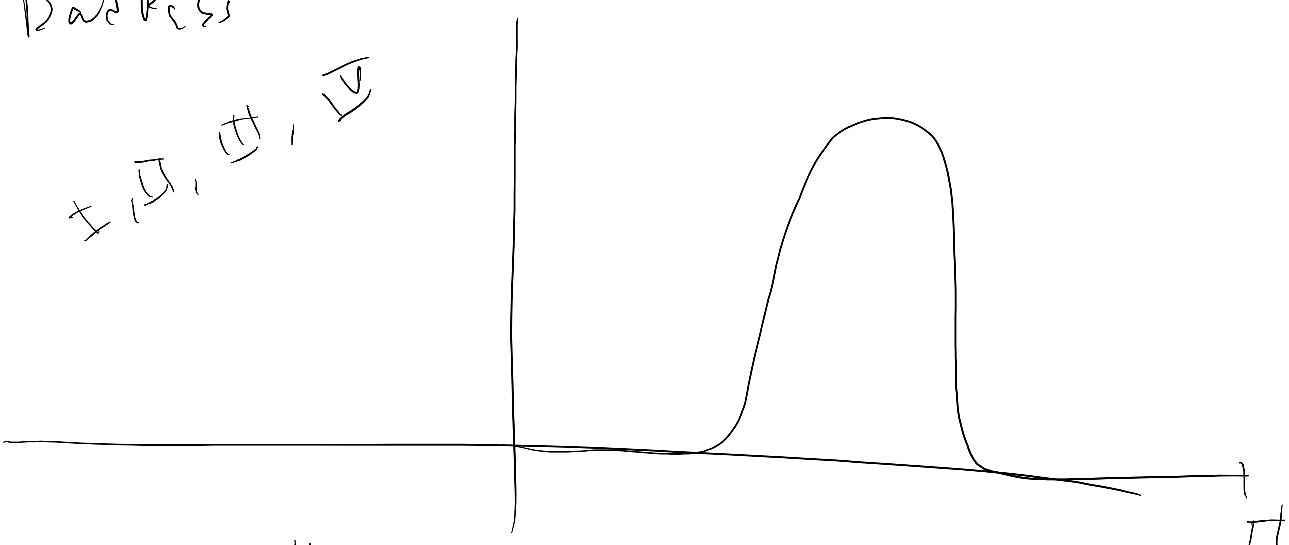
High pass Filter



Type I & IV

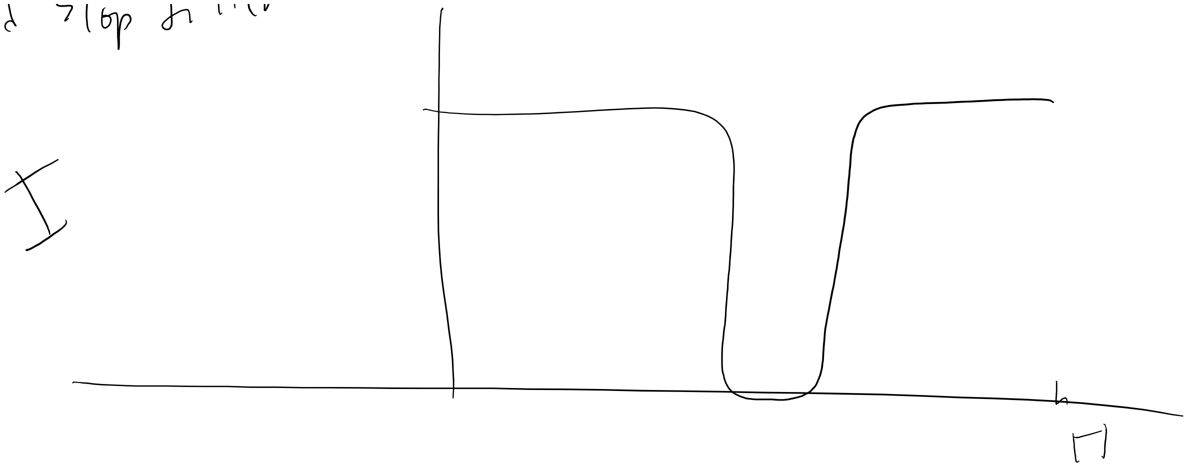
Band Pass

I, II, III, IV



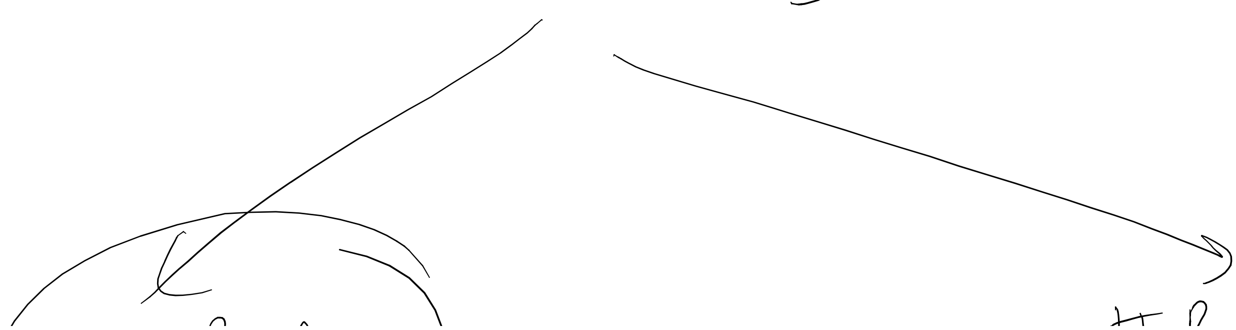
Band stop filter

Band Stop filter



	LP	HP	BP	BS
I	✓	✓	✓	✓
II	✓	X	✓	X
III	X	X	✓	X
IV	X	✓	✓	X

Filter Design
LTI



FIR. The
Determine coeff. of
h(n)

IIR.

Rational
Transfer fn. $H(z) = \frac{P(z)}{Q(z)}$
determine
coeffs of $P(z)$
and $Q(z)$

Diff. eqn.

not a rational
xfer fn

Build Filter

1. Specification → Application dependent

2. Design ← Course

3. Implementation — C prog on P.C

→ matlab

→ prog DSP chip
T.I.
A.D.

A.D.

FPGA /

Design ASIC

ASIC

FIR

Windows

Optimum.

FIR Filter Design Using Windows

(1) Start with desired freq. Respon $H_d(\omega)$

(2) Compute IDTFT $\{ H_d(\omega) \} = h_d(n) =$

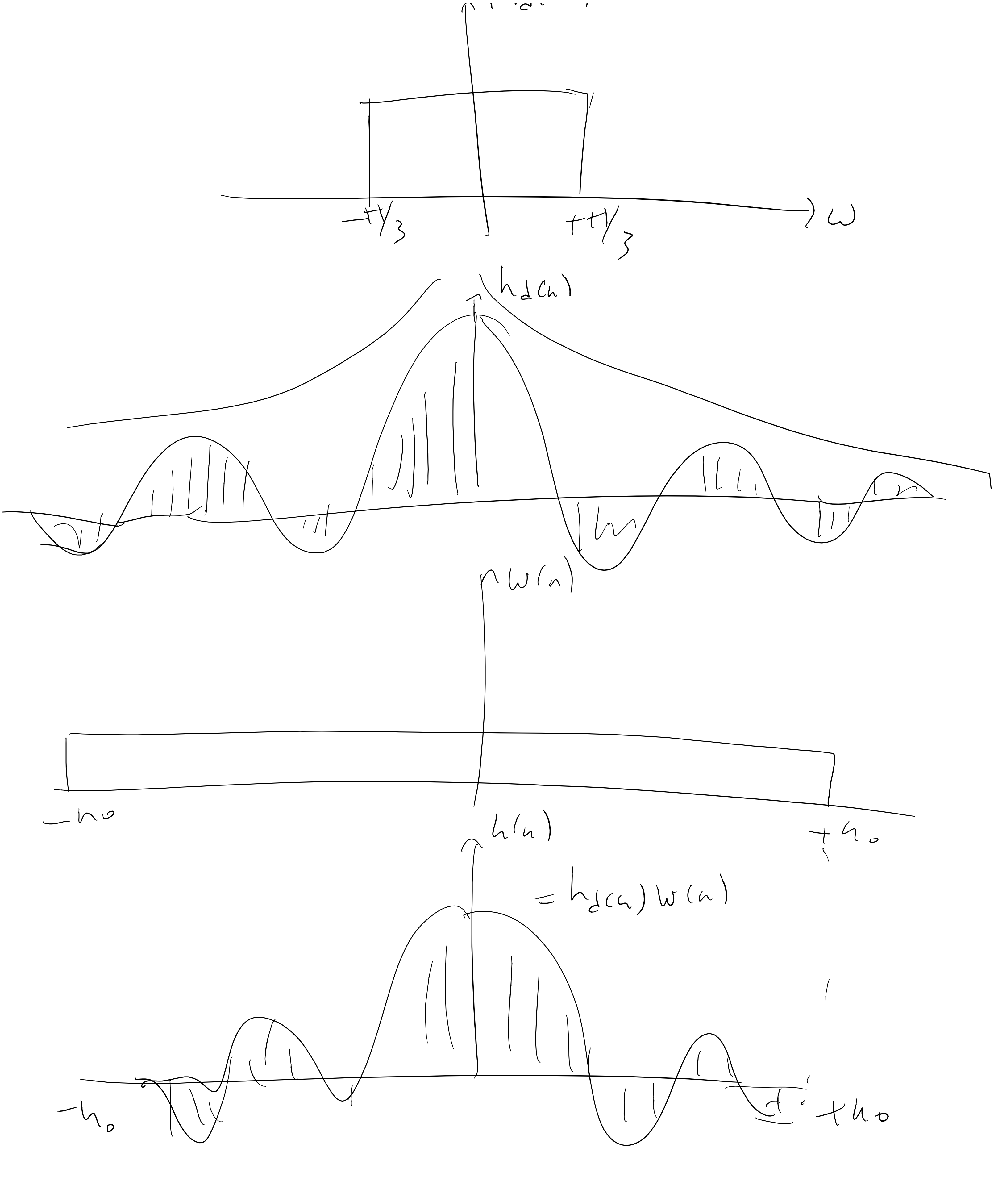
= desired Impulse respon.

(3) $h(n) = h_d(n)\, w(n)$

finite length window Function

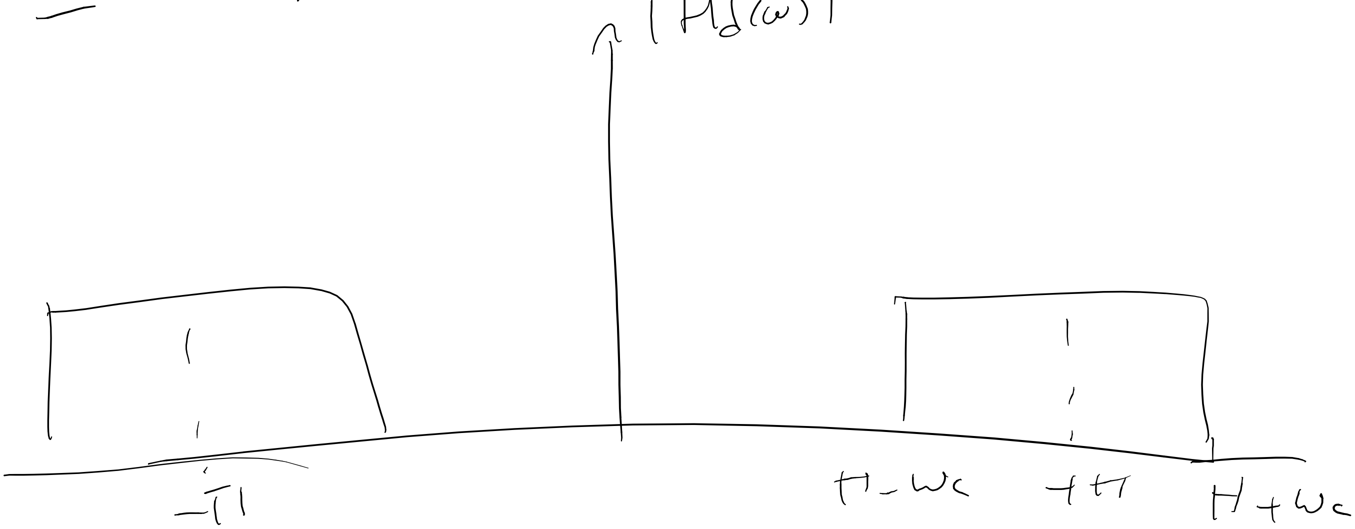
Ex

$H_d(\omega)$



-π/3
π/3
ω
hd(n)
w(n)
-n0
+n0
h(n)
= hd(n) w(n)
-n0
+n0

Ex High Pass Filter



(2) IDTFT $\{ H_d(\omega) \} =$

Assume Linear phase for both desired & final

Assume Type I $\left.\begin{array}{l}\beta = 0 \\ \alpha = \frac{N-1}{2}\end{array}\right\} \Longrightarrow$

$$\Longrightarrow H_d(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} e^{-j\alpha\omega}$$

$$|H_d(\omega)| = \begin{cases} 1 & \pi-\omega_c \le \omega < \pi+\omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & \pi-\omega_c < \omega < \pi+\omega_c \\ 0 & \text{otherwise} \end{cases}$$

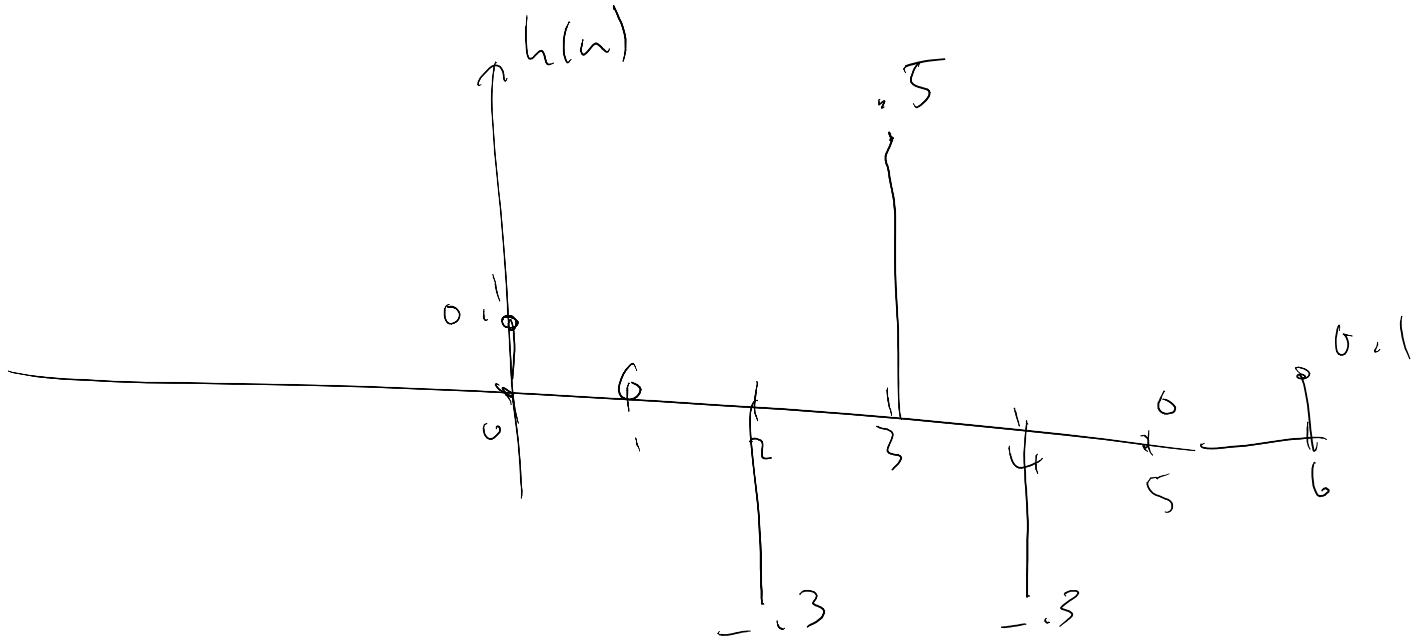
$$H_d(\omega) = \begin{cases} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{IDFT}\{H_d(\omega)\} = \frac{1}{2\pi} \int_{\pi - \omega_c}^{\pi + \omega_c} e^{-j\alpha\omega} e^{j\omega n} \, d\omega$$

$$h_d(n) = \frac{(-1)^{n-\alpha}}{\pi(n-\alpha)} \sin[\omega_c(n-\alpha)]$$

3) $h(n) = h_d(n) \, w(n)$

$w(n) \rightarrow 7$ Taps



$$[H(\omega)]_{\omega=\pi} = \sum_n h(n) e^{-j\pi n}$$

$= 0.1 - 0.3 - 0.5 - 0.3$

$= -0.9$

$$\left[H(\omega)\right]_{\omega=0} = 0.1$$